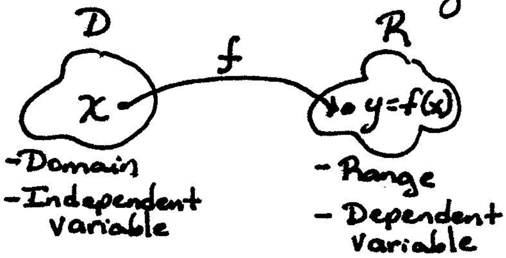
Summer 2006 Tomastik Calculus: Applications & Tech, 3e

1.1 Functions p. 5

Definition: Let D and R be two nonempty sets. A function f from D to R is a rule that assigns to each element x in D one and only one element $y = f(x)$ in R.



Finding the Domain of a Function and Evaluate

* Polynomial:

Domain → $(-\infty, \infty)$ & $\mathbb{R}$

Evaluate → $f(2)$, $f(1+x)$ using $f(x) = x^2 + 2x - 3$

Ex.

$f(2) = (2)^2 + 2(2) - 3 = 4 + 4 - 3 = 5$

$f(1+x) = (1+x)^2 + 2(1+x) - 3$

$= 1 + 2x + x^2 + 2 + 2x - 3 = x^2 + 4x$

Rational:

Domain → denominator $\neq 0$ "What values of x causes this?"

Evaluate → $f(-1)$, $f(x+2)$ using $f(x) = x+1/x^2$

Domain → $x^2 \neq 0$ $x \neq 0$: $(-\infty, 0) \cup (0, \infty)$; All real #'s except $x = 0$.

Ex.

$f(-1) = (-1)+1/(-1)^2 = 0/1 = 0$

$f(x+2) = (x+2)+1/(x+2)^2 = x+3/x^2+4x+4$

Radical:

Domain → radicand ≥ 0 $\qquad f(x)=\sqrt{\text{radicand}}$

$f(x)=\sqrt{3x-11}$

Domain: $3x-11 \geq 0$

$3x \geq 11$

$x \geq 11/3$ or $[11/3, \infty)$

Ex. Evaluate: $f(5)$, $f(1)$, $f(t)$

$f(5)=\sqrt{3(5)-11} = \sqrt{15-11} = \sqrt{4} = 2$

$f(1)=\sqrt{3(1)-11} = \sqrt{-8}$ ← Not a $\mathbb{R}$

$f(t)=\sqrt{3t-11}$

Combinations:

$f(x)=\dfrac{9x^2-5}{x^2-9}$ $\qquad x^2-9 \neq 0 \qquad x \neq \pm 3$

$x^2 \neq 9 \qquad *(-\infty,-3)\cup(-3,3)\cup(3,\infty)*$

Ex. $f(x)=\dfrac{x \quad \mathbb{R}}{\sqrt{x-3}}$ $\qquad x-3>0$

$x>3 \qquad *(3,\infty)*$

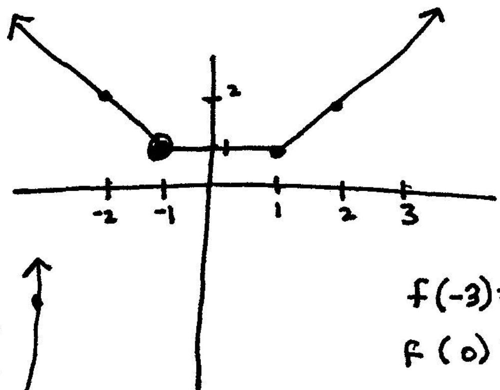
$f(x)=\dfrac{\sqrt{x+1}}{x^2-x-6}$ $\Rightarrow x+1 \geq 0 \qquad x \geq -1$

$\Rightarrow (x-3)(x+2) \neq 0 \qquad x \neq 3,-2 \qquad *[-1,3)\cup(3,\infty)*$

Piece-wise Functions:

Ex.

$$f(x)=\begin{cases} -x & \text{if } x \leq -1 \quad \text{(Blue)} \\ 1 & \text{if } -1 < x < 1 \quad \text{(Red)} \\ x & \text{if } x \geq 1 \quad \text{(Green)} \end{cases}$$



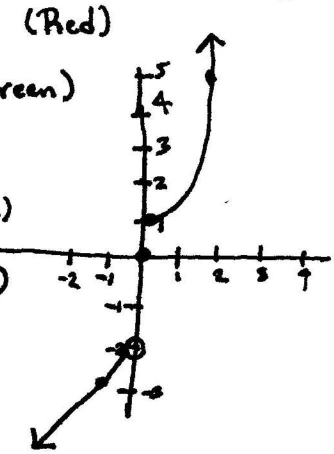
Continuous $(-\infty,\infty)$

$f(-3)=$

$f(0)=$

$f(7)=$

$$f(x)=\begin{cases} x^2+1 & x \geq 0 \quad \text{(Red)} \\ x-2 & x < 0 \quad \text{(Blue)} \end{cases}$$



Discontinuous @ $x=0$

$f(2)= \qquad f(11)=$

$f(-10)=$

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1.2 Mathematical Models p. 24

Mathematical Models of Cost, Revenue, & Profit

Two types of cost : variable and fixed.

Cost

$C(x) = mx + b$: Linear   Ex. #2 $C(x) = 6x + 14{,}000$

variable   fixed : $C(x) \geq 0$

Revenue

$R(x) = $ (price per unit) x (number sold)

$R(x) = px$: Linear   Ex. #4 $R(x) = 0.1x$

: $R(x) \geq 0$

Profit

Profit = Revenue − Cost : Linear or not

$P(x) = R(x) - C(x)$ $P(x)$: +/− $-P(x)$: loss

Ex. #6 $-5.9x - 14{,}000$

Break-Even Quantity $x = $___

① $P(x) = 0$

② $R(x) - C(x) = 0$

③ $R(x) = C(x)$

Ex. #8 $C = 3x + 10$, $R = 6x$

Break-even quantity:

$P(x) = R(x) - C(x) = 0$

$P(x) = 6x - (3x+10) = 0$

$= 6x - 3x - 10$

$P(x) = 3x - 10 = 0$

$P(x) = 0$

$3x - 10 = 0$

$x = \frac{10}{3} = 3\frac{1}{3}$ 4

$R(x) = C(x)$

$6x = 3x + 10$

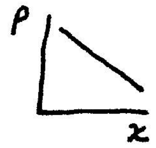
$3x = 10$

$x = \frac{10}{3}$

$x = 3\frac{1}{3}$ 4

p. 31 Mathematical Models of Supply and Demand

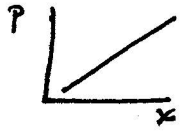
Typical demand curve $p = -cx + d$



p: price of the x^{th} unit sold

x: # of units produced or sold by the entire industry during a given period of time.

Typical supply curve $p = cx + d$



p: price necessary for suppliers to make available x units to the market

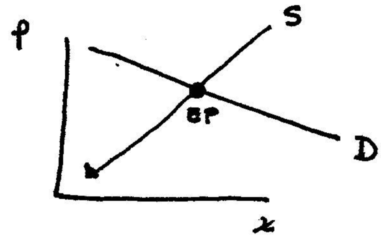
x: # of units supplied

"Supplier of any product want to sell more if the price is higher."

The point of intersection or the point at which supply equals demand: equilibrium point (x, y) or $(x, p(x))$

x → equilibrium quantity

$p(x)$ → equilibrium price



Example

Sketch the supply and demand curve and find the equilibrium point.

Demand: $-0.1x + 2 = p$

Supply: $0.2x + 1 = p$

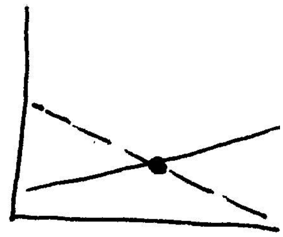
Alg: $-0.1x + 2 = 0.2x + 1$

$1 = 0.3x$

$10 = 3x$

$x = \frac{10}{3} = 3\frac{1}{3}$ 4

Graph:



$x = 3.333$

$x = 3\frac{1}{3} = \frac{10}{3}$

$\boxed{x = 4}$

 Quadratic Models

$f(x) = ax^2 + bx + c$

If $a > 0$, ∪ - concave up.

If $a < 0$, ∩ - concave down.

Vertex: $\left(x = \frac{-b}{2a}, y = f\left(\frac{-b}{2a}\right)\right)$

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

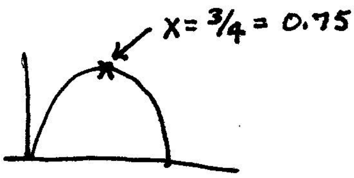
Where either occurs | maximum value if $a < 0$ | minimum value if $a > 0$

<u>Examples</u>: Find <u>where</u> the revenue function is <u>maximized</u> using the given linear function: $p = -2x + 3$

Using $R = xp$, $R = x(-2x+3) = -2x^2 + 3x$ (quadratic) ∩ $a < 0$

Alg: $x = \frac{-b}{2a} = \frac{-3}{2(-2)} = \frac{-3}{-4} = \frac{3}{4}$

Graph:



Given $R(x) = -3x^2 + 20x$ and $C(x) = 2x + 15$, find

a. profit function: $P(x) = R(x) - C(x) = -3x^2 + 20x - (2x + 15)$

$= -3x^2 + 20x - 2x - 15 = -3x^2 + 18x - 15$ (quadratic)

b. x-value that maximized the profit.

$x = \frac{-b}{2a} = \frac{-18}{2(-3)} = 3$

c. break-even quantities, if the exist: $P(x) = 0$

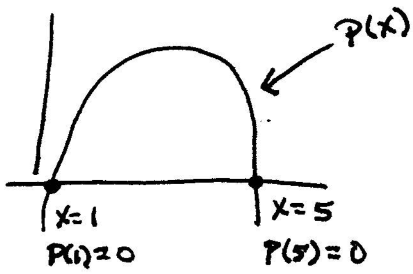
<u>Alg.</u> $-3x^2 + 18x - 15 = 0$

$-3(x^2 - 6x + 5) = 0$

$-3(x-5)(x-1) = 0$

$x = 5 \quad x = 1$

<u>Graph</u>



Compound Interest: $A = P(1+\frac{r}{n})^{nt}$

A: amount resulting

P: principal

r: rate (decimal)

n: # of times compounded per year

t: # of years

Effective Yield: $r_{eff} = (1+\frac{r}{n})^{n} - 1$

Present Value:

$P = A(1+\frac{r}{n})^{-nt}$

$\downarrow$

present value needed currently in an account so that a future amount, A, will be obtained

Examples

#40, (a)-(e)

#44, (a)-(e)

#46, (a)-(c)

Continuous Compounding

$A = Pe^{rt}$

A: amount resulting

P: principal

r: rate (decimal)

t: # of years

Effective Yield

$r_{eff} = e^{r} - 1$

Present Value

$P = Ae^{-rt}$

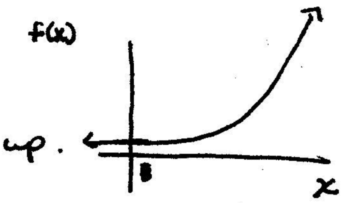
Examples

#40 (f)

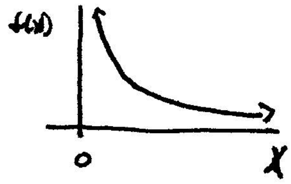
#44 (f)

#46 (d)

* Properties of the exponential function $y = f(x) = a^x$



Property 1. If $a > 1$, the function a^x is increasing and concave up.



Property 2. If $0 < a < 1$, the function a^x is decreasing and concave up.

Property 3. If $a \neq 1$, then $a^x = a^y$ if and only if $x = y$.

Property 4. If $0 < a < 1$, then the graph of $y = a^x$ approaches the x-axis as x becomes large $(y = a^{\infty} \to 0)$

Property 5. If $a > 1$, then the graph of $y = a^x$ appoaches the x-axis as x becomes negatively large. $(y = a^{-\infty} \to 0)$

* Some Definitions Involving Exponentials

$a^0 = 1$ $\quad a^{1/2} = \sqrt{a}$ $\quad a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$

$a^{-1} = \frac{1}{a}$ $\quad a^{1/3} = \sqrt[3]{a}$ $\quad \sqrt[n]{a^m}$

$a^{-x} = \frac{1}{a^x}$ if $a \neq 0$ $\quad a^{1/n} = \sqrt[n]{a}$ $\quad (\sqrt[n]{a})^m$

* Further Properties of Exponentials

Property 6. $a^x \cdot a^y = a^{x+y}$

Property 7. $\frac{a^x}{a^y} = a^{x-y}$ $\qquad a^x \cdot a^{-y} = a^{x+(-y)} = a^{x-y}$

Property 8. $(a^x)^y = a^{xy}$

Examples

① Graph without/with calculator.

$a > 1$; $a = 2$

$y = 2^x$

$0 < a < 1$; $a = 1/2$

$y = (1/3)^x$

② Solve for x.

$(1/2)^x = \frac{1}{16}$

$5^{3x} = 125^{4x-4}$

③ Solve for x.

$2^{5x} = 2^{x+8}$

$5^{2x-1}\,5^x = \frac{1}{5^x}$

$x^2\,7^x = 7^x$

$(x^2 - 3x - 10)4^x = 0$

SSI 1.4 Combinations of Functions

Let f and g be two functions and define

$$D = \{x \mid x \in (\text{domain of } f) \text{ and } x \in (\text{domain of } g)\}$$

Then for all $x \in D$ we define

$f+g$ (sum) $= (f+g)(x) = f(x) + g(x)$

$f-g$ (difference) $= (f-g)(x) = f(x) - g(x)$

$f \cdot g$ (product) $= (f \cdot g)(x) = f(x) \cdot g(x)$

f/g (quotient) $= \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ Note: $g(x) \neq 0$

$f(x) =$

$g(x) =$

	Domain (Alg)	Domain (Graph.)
$(f+g)(x) =$		
$(f-g)(x) =$		
$(f \cdot g)(x) =$		
$\left(\frac{f}{g}\right)(x) =$		
$\cancel{(f \circ g)(x) =}$		

The Composition of a Function

$h(x) = f[g(x)]$

Examples

$h(x) = (x^2 + 2x - 11)^5$

"simplest" $f(x) =$

$g(x) =$

$h(x) = \sqrt[3]{2x^4 - 31}$

"simplest" $f(x) =$

$g(x) =$

Definition of a Composite Function

Let f and g be two functions. The composite function $f \circ g$ is defined by

$$(f \circ g)(x) = f[g(x)].$$

The domain of $f \circ g$ is the set of a x in the domain of g for which $g(x)$ is in the domain of f.

Example

$f(x) =$

$g(x) =$

Domain:

Domain:

Domain:

$(f \circ g)(x) =$

$(g \circ f)(x) =$

$(g \circ g)(x) =$

$(f \circ f)(x) =$

SSI06 1.5 Logarithms p. 78

Definition: Logarithm to the Base 10

If $x>0$, the logarithm to the base of 10 of x, denoted by $\log_{10} x$, is defined as follows

$$y=\log_{10} x \quad \text{if and only if} \quad x=10^y$$

$$\log_{10} x = y \quad \text{if and only if} \quad 10^y = x$$

x: "answer"

y: exponent

$\log_{\ }$: base

Definition: Logarithm to the Base a

Let a be a positive number with $a \neq 1$. If $x>0$, the logarithm to the base a of x, denoted by $\log_a x$, is defined as follows

$$y=\log_a x \quad \text{if and only if} \quad x=a^y$$

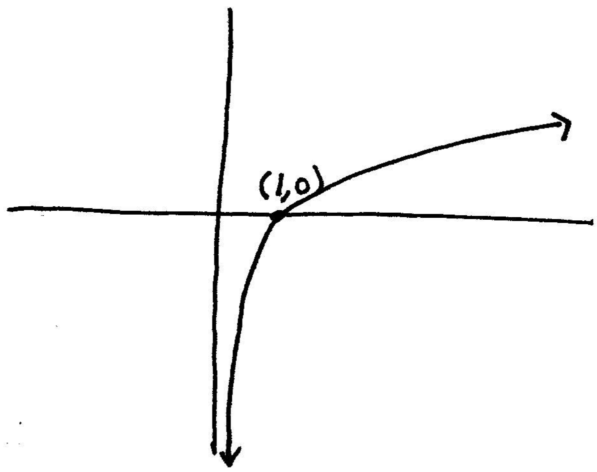
$$\log_a x = y \qquad\qquad a^y = x$$

Basic Properties of the Logarithmic Function

Property 1. $a^{\log_a x} = x$ if $x>0$

Property 2. $\log_a a^x = x$ for all x

Graph. (p. 81) / Properties $a>1$



Domain: $(0,\infty)$

Range: $(-\infty,\infty)$

Increasing: $(0,\infty)$

Concave down: $(0,\infty)$

$\log x = \log y \Leftrightarrow x=y$

$\ln x = \ln y \Leftrightarrow x=y$

For any positive $a \neq 1$ the logarithm $\log_a x$ obey the following rules

Rule 1, $\log_a xy = \log_a x + \log_a y$

Rule 2. $\log_a \frac{x}{y} = \log_a x - \log_a y$

Rule 3, $\log_a x^c = c \log_a x$

Definition <u>Natural</u> <u>Logarithm</u>

If $x > 0$, the natural logarithm of x, denoted by $\ln x$, is defined as follows:

$$y = \ln x \quad \text{if and only if} \quad x = e^y$$

$$y = \ln x = \log_e x$$

<u>Change of Base Theorem</u>

$$\log_a x = \frac{\log_b x}{\log_b a} = \underset{\substack{\uparrow \\ b=e}}{\frac{\ln x}{\ln a}} = \underset{\substack{\uparrow \\ b=10}}{\frac{\log x}{\log a}}$$

Ex. $\log_{11} 25 =$

Examples

1. Solve for x.

$\log x = -3$ $\qquad$ $\ln x = -3/4$ $\qquad$ $2\log_2 x = 9$

2. Simplify.

$\log\sqrt{10}$ $\qquad$ $e^{0.5\ln 9}$ $\qquad$ $\sqrt{3^{\log_3 2}}$

3. Rewrite in terms of $\log x$, $\log y$ and $\log z$

$$\log\frac{x^2y^3}{\sqrt{z}}$$

4. Rewrite as one logarithm.

$$2\log x - \tfrac{1}{2}\log y + \log z$$

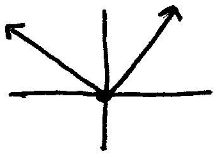
5. Solve each equation for x.

$2\cdot 10^{3x-1} = 5$ $\qquad$ $e^{\sqrt{x}} = 4$ $\qquad$ $\log x^2 = 2$ $\qquad$ $\log_3 7x = \log_3 10$

$3\log(x+1) + 1 = 0$ $\qquad$ $3\log(x+1) - 6 = 0$

Know the basic graphs.

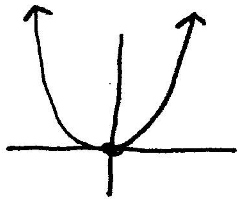
$y = |x|$



Decreasing: $(-\infty, 0)$

Increasing: $(0, \infty)$

$y = x^2$



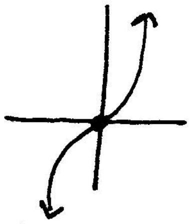
Decreasing: $(-\infty, 0)$

Increasing: $(0, \infty)$

Concave up: $(-\infty, \infty)$

Concave down: —

$y = x^3$



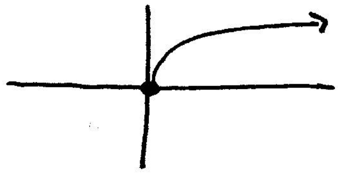
Decreasing: —

Increasing: $(-\infty, \infty)$

Concave up: $(0, \infty)$

Concave down: $(-\infty, 0)$

$y = \sqrt{x}$



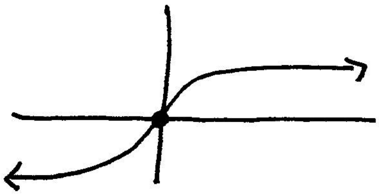
Decreasing: —

Increasing: $(0, \infty)$

Concave down: $(0, \infty)$

Concave up: —

$y = \sqrt[3]{x}$



Decreasing: —

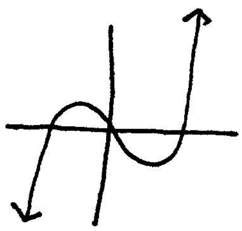
Increasing: $(-\infty, \infty)$

Concave up: $(-\infty, 0)$

Concave down: $(0, \infty)$

Vertical Line Test

1



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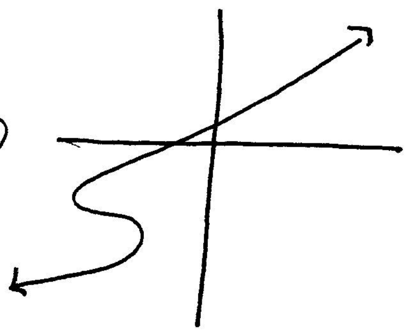

Continuous / Increasing / Decreasing / concavity (Graph)